Symmetric matrices — Tutorial — 1

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Exercise 1. ***Bilinear symmetric / quadratic form from their matrix***
For each of the following square matrices, provide the expressions of the bilinear symmetric form φ_i and of the quadratic form q_i, the matrix of which, in the canonical basis of $\mathbb{R}^n$, is A.

1. $A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$.
2. $A_2 = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & 4 \\ 3 & 4 & 0 \end{pmatrix}$.
3. $A_3 = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

Exercise 2. ***Polar form and change of basis***
Let us consider the function $q : \mathbb{R}^3 \to \mathbb{R}$ defined as: $q(x, y, z) = 4x^2 + 2y^2 + z^2 + 2xz - 2yz$.

1. Check that q is a quadratic form on $\mathbb{R}^3$, and provide its matrix in the canonical basis of $\mathbb{R}^3$.
2. Check that the three vectors $u = (1, -1, -2)$, $v = (0, 1, 1)$ and $w = (0, 0, 1)$ define a basis of $\mathbb{R}^3$, and provide, by two different methods, the matrix of q in this basis.
3. Is the polar form of q an inner product?

Exercise 3. ***Quadratic form on a space of polynomials and its matrix in bases***
We consider the map $\varphi : \mathbb{R}_3[X] \to \mathbb{R}_3[X]$, defined as:

$$\varphi(P, Q) = P(0)Q(0) + \int_0^1 P'(x)Q'(x)\, dx\,.$$

1. Prove that φ is an inner product on $\mathbb{R}_3[X]$.
2. Let us denote by the vector subspace of $\mathbb{R}_3[X]$ spanned by the polynomials 1 and X, and let us consider a basis $\mathcal{B}$ of $\mathbb{R}_3[X]$, of the form $\mathcal{B} = (1, X, Q, R)$, where Q and R belong to $F^{\perp}$.
   a) Explain why such a basis indeed exists.
   b) What can you say about the matrix $[\varphi]_{\mathcal{B}}$? (provide as much information as possible).
   c) Provide all the bases $\mathcal{B}$ of this form, with $\deg(Q) = 2$ and $\deg(R) = 3$ and

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

Exercise 4. 3×3 ***symmetric matrix***
Let us consider the matrix $A = \begin{pmatrix} 6 & 4 & 5 \\ 4 & 6 & 5 \\ 5 & 5 & 5 \end{pmatrix}$.

1. a) What is the rank of A. Deduce from this rank one eigenvalue of A, and its multiplicity.
   b) Show that 2 is an eigenvalue of A.
   c) Find the remaining eigenvalue of A.
2. Compute the characteristic polynomial of A, and recover its eigenvalues.
3. Provide a basis of eigenvectors of A, and diagonalize A in an orthonormal basis.
4. Is A the matrix of an inner product?

Exercise 5. 3×3 ***symmetric matrix, 2***

Let us consider the matrix $A = \begin{pmatrix} 7 & 2 & -2 \\ 2 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix}$.

1. Compute the eigenvalues of A, and diagonalize A in an orthonormal basis.
2. Is A the matrix of an inner product?

Exercise 6. 3×3 ***symmetric matrix, 3***

Let us consider the matrix $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$.

1. Compute the eigenvalues of A, and diagonalize A in an orthonormal basis (let us denote by $\mathcal{B}$ this orthonormal basis, and by $\mathcal{B}_{\text{can}}$ the canonical basis of $\mathbb{R}^3$).
2. Let us denote by φ the symmetric bilinear form on $\mathbb{R}^3$ with matrix A in the basis $\mathcal{B}_{\text{can}}$. Is this bilinear form an inner product?
3. Is $\mathcal{B}$ orthogonal/orthonormal for φ?
4. Provide a basis $\mathcal{B}'$ which is orthonormal for φ and orthogonal for the canonical dot product of $\mathbb{R}^3$.

Exercise 7. ***Matrix, rank, and signature of a quadratic form***

For each of the following quadratic forms on $\mathbb{R}^n$, provide its matrix in the canonical basis of $\mathbb{R}^n$, its signature, and its rank, and say whether the corresponding polar form is, or not, an inner product.

1. $n = 2$, $q(x) = x_1 \; x_2$
2. $n = 2$, $q(x) = 9x_1^2 \; + 2x_1x_2 - x_2^2$
3. $n = 3$, $q(x) = x_1^2 \; - x_2^2 - x_1 \; x_3$
4. $n = 3$, $q(x) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2$
5. $n = 3$, $q(x) = 2x_1^2 + 6x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 6 \; x_2x_3$
