Approx. ($\sim$ 5h) on each topic.

1 Numerical series

Exercise 1
Determine the nature of the series with general term u_n in each of the following cases :

1. $u_n = \dfrac{1}{n - 10n^3}$
2. $u_n = e^{1/n}$
3. $u_n = \dfrac{\ln n}{n}$
4. $u_n = e^{-\sqrt{\ln(n)}}$
5. $u_n = \dfrac{n^2}{2^n}$
6. $u_n = \dfrac{(n!)^2}{2^{n^2}}$
7. $u_n = \dfrac{n^2}{n^3+1}$
8. $u_n = n \sin\left(\dfrac{1}{n}\right)$
9. $u_n = \dfrac{n^n}{2^n}$
10. $u_n = \dfrac{1}{(\ln(n))^n}$
11. $u_n = \dfrac{1}{\ln(n^2+n+1)}$
12. $u_n = \ln\left(\dfrac{n^2+n+1}{n^2+n-1}\right)$
13. $u_n = \dfrac{2+\cos(n)}{n}$
14. $u_n = \dfrac{2+\cos(n)}{n^2}$
15. $u_n = 2^{\sqrt{n}}$
16. $u_n = \dfrac{1}{\sqrt{n(n+1)(n+2)}}$
17. $u_n = e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$
18. $u_n = \dfrac{\sin(n^2)}{n^2}$
19. $u_n = \dfrac{1}{(\ln(n))^{\ln(n)}}$
20. $u_n = \dfrac{\mathrm{e}^{\mathrm{i}n}}{n^{9/8}}$ où $\mathrm{i}^2 = -1$

Exercise 2
Consider a sequence $(u_n)_{n\in\mathbb{N}}$ of strictly positive real numbers such that the series $\sum u_n$ converges. What can be said about the series with general term $\dfrac{u_n}{1+u_n}, e^{u_n}, u_n^2$ et $\sqrt{u_n}$?

Some series for which the sum can be computed

Exercise 3
Compute the sum of the series with general term u_n in each of the following cases :

1. $u_n = \displaystyle\int_n^{n+1} \frac{\mathrm{d}t}{1+t^2} (n \geq 1)$
2. $u_n = \dfrac{1+n}{n!} (n \geq 1)$
3. $u_n = \dfrac{1}{(4-(-1)^n)^n}$
4. $u_n = \dfrac{1}{(2n+1)^2}$ (sachant que $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)
5. $u_n = \dfrac{1}{4n^2-1}$
6. $u_n = \ln\left(1 - \dfrac{1}{n^2}\right)$ $(n \geq 2)$
7. $u_n = \dfrac{n+1}{3^n}$

Exercise 4
For any $n \in \mathbb{N}$, define $n \in \mathbb{N}, u_n = \dfrac{(-1)^{n+1}}{n}$.

1. (a) Show that for any $N \geq 1, \displaystyle\sum_{n=1}^{N} u_n = \int_0^1 \frac{1-(-x)^N}{1+x}\,\mathrm{d}x$.

   (b) Show that $\displaystyle\lim_{N\to+\infty} \int_0^1 \frac{x^N}{1+x}\,\mathrm{d}x = 0$.
2. Deduce the value of $S = \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$.

Alternating Series

Exercise 5
Let the series $\displaystyle\sum_{n\geq 2} \frac{(-1)^n}{\ln(n)}$. Using the Alternating Series Test, show that the series converges and determine a rank N from which the remainder R_N is less than or equal to 10^{-1}.

Exercise 6
Show that the series $\sum_{n\geq 1} \frac{(-1)^n \ln(n)}{n}$ converges.

Error Control in the Approximation of the Sum of a Series

Exercise 7

1. Establish the convergence of the numerical series with general term $u_n = \frac{1}{n^n}$.
2. For all $n \in \mathbb{N}^*$, let $R_n = \sum_{k=n+1}^{+\infty} \frac{1}{k^k}$ denote the remainder of order n for the series with general term u_n. By comparing R_n to a geometric series, establish the bound $R_n \leq \frac{1}{n(n+1)^n}$ for all $n \in \mathbb{N}^*$.
3. Compute the sum S of the series with general term $u_n = \frac{1}{n^n}$ with an error not exceeding 10^{-3}.

Exercise 8
Let $u_n = \frac{1}{(2n-1)5^{2n-1}}$ with $n \geq 1$.

1. Show that the series $\sum u_n$ converges.
2. Let $R_n = \sum_{k=n+1}^{+\infty} u_k$. Show that $R_n \leq \frac{1}{24} u_n$ and deduce the value of $\sum_{k=1}^{+\infty} u_k$ to within 0.001.

Exercise 9 (True/False to Test Yourself)
For each of the following statements, determine whether it is true or false. Justify your answer.

1. If for all n, $\left|\frac{u_{n+1}}{u_n}\right| < 1$, then $\sum u_n$ converges.
2. If $\sum u_n$ converges, then $\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$.
3. If for all n, $\left|\frac{u_{n+1}}{u_n}\right| > 1$, then $\sum u_n$ diverges.
4. If for all $n, u_n \leq v_n$ and if $\sum v_n$ converges, then $\sum u_n$ converges.
5. If $\sum u_n$ converges, then $\sum u_n^2$ converges.
6. If $\lim_{n\to+\infty} nu_n = 2$, then $\sum u_n$ diverges.
7. If $\lim_{n\to+\infty} nu_n = 0$, then $\sum u_n$ converges.
8. If $|u_n| \geq \frac{1}{n}$, then $\sum u_n$ diverges.
9. If for all $n, u_n > 0$ and if $\sum u_n$ converges, then the sequence (u_n) decreases from a certain rank onward.

2 Power series

Exercise 10 (Computation of Radii of Convergence)
Determine the radius of convergence of the following power series :

1. $\sum \frac{(n!)^2}{(2n)!} z^n$
2. $\sum_{n\geq 1} (-1)^n \frac{2^n}{n+n^2} z^n$
3. $\sum \frac{z^{2n}}{2^n}$
4. $\sum_{n\geq 1} \frac{(1+\mathrm{i})^n}{n2^n} z^n$
5. $\sum a^{\sqrt{n}} z^n, a > 0$
6. $\sum z^{n!}$
7. $\sum_{n\geq 3} \tan\left(\frac{\pi}{n}\right) z^n$
8. $\sum_{n\geq 1} \left(1-\frac{1}{n}\right)^{n^2} z^n$
9. $\sum_{n\geq 1} \cos\left(\frac{1}{n}\right) z^n$
10. $\sum \cos(n) z^n$
11. $\sum_{n\geq 2} (-1)^n \frac{z^n}{\ln n}$
12. $\sum_{n\geq 1} (\ln n) z^n$
13. $\sum n! z^{n^2}$
14. $\sum \frac{z^n}{(3+(-1)^n)^n}$
15. $\sum \left(\sin\left(\frac{n\pi}{3}\right)\right)^n z^n$
16. $\sum \frac{n!}{(2n)!} z^n$

Exercise 11
In the following cases, what can be said about the radius of convergence of the power series $\sum a_n z^n$?

1. $\sum a_n$ converges et $\sum (-1)^n a_n$ diverges.
2. $\forall n \in \mathbb{N}, |a_n| < 2^n$.
3. $\sum \frac{(-1)^n n a_n}{4^n}$ diverges.
4. $\forall n \in \mathbb{N}, \frac{1}{2n+1} \leq a_n \leq \frac{1}{n+1}$.

Exercise 12

1. Recall the radius of convergence of the power series $\sum z^n$, $\sum_{n\geq 1} \frac{z^n}{n}$, and $\sum_{n\geq 1} \frac{z^n}{n^2}$.
2. Study the convergence of $\sum z^n$ and $\sum_{n\geq 1} \frac{z^n}{n^2}$ on the boundary of the disk of convergence.
3. Examine the continuity of the real-variable functions $f : x \mapsto \sum_{n\geq 0} x^n$ and $g : x \mapsto \sum_{n\geq 1} \frac{x^n}{n^2}$.
4. Analyze the convergence of the power series $\sum_{n\geq 1} \frac{z^n}{n}$ at the endpoints of the real segment of convergence.

Exercise 13
Describe the general shape of the curves representing the following functions. Do not attempt to simplify the expression of the function, but use local approximations with Taylor formulas.

1. $f : x \longmapsto \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n+1}} x^{2n}$
2. $g : x \longmapsto \sum_{n=0}^{+\infty} \frac{n+1}{9^n} x^{2n+1}$

Exercise 14
For each of the following power series, determine its real interval of convergence and its sum.

1. $\sum \frac{3n+2}{n!} x^n$
2. $\sum_{n\geq 1} \frac{(-1)^n x^{2n+1}}{n}$
3. $\sum \frac{(-1)^n x^{2n+1}}{(2n)!}$
4. $\sum_{n\geq 1} \frac{1+4^n}{n} x^n$
5. $\sum n^2 x^n$
6. $\sum \frac{(3x)^n}{n^2+3n+2}$
7. $\sum \frac{n x^n}{(4+(-1)^n)^n}$

8. $\sum \dfrac{x^n}{(2n+1)!}$

9. $\sum_{n\geq 1} \dfrac{x^{n+1}}{n(n+1)}$

10. $\sum nx^{2n}$

Power Series Expansion

Exercise 15
Determine the power series expansion of the following functions, specifying the open interval of convergence.

1. $f(x) = \dfrac{1}{(1+x)^2}$
2. $f(x) = (1+x)\mathrm{e}^{-x}$
3. $f(x) = \dfrac{2x-1}{(x-1)(x-2)}$
4. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$
5. $f(x) = \mathrm{e}^x \cos(x)$
(use a complex exponential)

Exercise 16
Let f be the function defined on $]-1, 1[$ by $f(x) = \dfrac{\arcsin(x)}{\sqrt{1-x^2}}$.

1. Justify that f can be expanded as a power series on $]-1, 1[$.
2. Show that f satisfies the differential equation $\left(1-x^2\right) y'(x) - xy(x) = 1$.
3. Determine the power series expansion of f on $]-1, 1[$.

Exercise 17 (Study of a Iterative Sequence)
Consider the sequence (a_n) defined by $a_0 = 2$ and for all $n \geq 1$,

$$2na_n = a_{n-1} + \frac{1}{(n-1)!}.$$

1. Derive a first-order differential equation satisfied by $f : x \mapsto \sum_{n=0}^{+\infty} a_n x^n$ on its open domain of convergence.
2. Solve this equation and deduce an explicit expression for a_n as a function of n.

Exercise 18 (Solving Differential Equations)
Consider the differential equation $(E) : x^2 y''(x) - 2xy'(x) + \left(2 - x^2\right) y(x) = 0$.

1. Show that (E) admits at least one solution expandable as a power series in the neighbourhood of 0.
2. Compute the radius of convergence of the resulting series.
3. Express the sum of the power series solution of (E) in terms of standard functions.
4. Add the initial conditions $y'(0) = 1$ and $y''(0) = 0$. What are the solutions of (E) under these conditions ?
5. Same question with $y(0) = 1$ and $y'(0) = 0$.

Exercise 19
Consider the following differential equation : $(E) : xy'(x) - y(x) = x^2 \mathrm{e}^{x^2}$. Find all solutions of (E) expandable as a power series and satisfying

$$\begin{cases} y(0) = 0 \\ y'(0) = a \end{cases} \qquad (a \in \mathbb{R}).$$