Inner products — Tutorial

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Exercise 1. ***Definitions***

1. For each of the following maps, say whether or not it is a symmetric bilinear form / an inner product.
   a) $\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto 2u_1v_2 - 3u_2 v_1$.
   b) $\varphi : \mathbb{R}[X] \times \mathbb{R}[X] \to \mathbb{R}$, $(P,Q) \mapsto P(0) + Q(0)$.
   c) $\varphi : \mathcal{C}^0\big([0,1],\mathbb{R}\big) \times \mathcal{C}^0\big([0,1],\mathbb{R}\big) \to \mathbb{R}$, $(f,g) \mapsto \displaystyle\int_0^1 t^2 f(t)g(t)\,dt$.
   d) $\varphi : \mathcal{C}^0\big([0,2],\mathbb{R}\big) \times \mathcal{C}^0\big([0,2],\mathbb{R}\big) \to \mathbb{R}$, $(f,g) \mapsto \displaystyle\int_0^2 (1-t) f(t)g(t)\,dt$.
   e) $\varphi : \mathcal{C}^0\big([0,2],\mathbb{R}\big) \times \mathcal{C}^0\big([0,2],\mathbb{R}\big) \to \mathbb{R}$, $(f,g) \mapsto \displaystyle\int_0^1 (1-t) f(t)g(t)\,dt$.
   f) $\varphi : \mathcal{C}^1\big([-1,1],\mathbb{R}\big) \times \mathcal{C}^1\big([-1,1],\mathbb{R}\big)$, $(f,g) \mapsto f(0)g(0) + \displaystyle\int_a^b f'(t)g'(t)\,dt$, for some real quantities a, b satisfying $-1 \leq a < b \leq 1$.
2. Give an example of a non-symmetric bilinear form on $\mathbb{R}[X]$.
3. Let n and p denote a positive integer and a nonnegative integer, respectively. Under which condition on p does the map $\varphi : (P,Q) \mapsto \displaystyle\sum_{k=0}^{p} P(k)Q(k)$ define an inner product on $\mathbb{R}_n[X]$?

Exercise 2. ***A general way to define a quadratic form***

Let f and g be two linear form on a real vector space E. Prove that the function $q : E \to \mathbb{R}$, $u \mapsto f(u)g(u)$ defines a quadratic form on E.

Conversely, provide an example of a real vector space E and a quadratic form on E which is not of this form (that is, which cannot be written as the product of two linear forms).

Exercise 3. ***Applications of Cauchy–Schwarz inequality***

1. Prove that, for every function f in $\mathcal{C}^0\big([a,b],\mathbb{R}\big)$, the following inequality holds:

$$\left(\int_a^b |f(t)|\;dt\right)^2 \leq (b-a)\int_a^b |f(t)|^2\;\;dt\,.$$

For which function is this inequality actually an equality?

2. a) Find the least value of the quantity $(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$, over all possible values of the positive quantities x and y and z.
   b) Extend this result to an arbitrary number (instead of three) positive quantities.

Exercise 4. ***The orthogonal complement of the orthogonal complement of a subspace is this subspace (finite dimension)***

Let E denote an Euclidean vector space and F denote a vector subspace of E. Prove that $\big(F^{\perp}\big)^{\perp} = F$.

Exercise 5. ***Euclidean distance to a vector subspace***

In $\mathbb{R}^4$ equipped with the canonical dot product, let $F = \operatorname{span}\big((1,2,-1,1),(0,3,1,-1)\big)$.

1. a) Find a system of equations of the coordinates (x,y,z,t) of a vector of $\mathbb{R}^4$, such that $F^{\perp}$ is equal to the solutions of this system.
   b) Provide an orthonormal basis of $F^{\perp}$.
2. Let $e_1 = (1,0,0,0)$. Compute the distance between e_1 and F.

Exercise 6. ***Euclidean distance to a vector subspace, 2***
In $\mathbb{R}^4$ equipped with the canonical dot product, let

$$u = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad F = \mathrm{span}(v_1, v_2, v_3)\,, \quad \text{with} \quad v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

1. What is the orthogonal subspace of u?
2. a) Provide an equation for F.
   b) Find the orthogonal projection of u on F and the distance between u and F.

Exercise 7. ***Trigonometric functions***
Let us consider the vector space $E = \mathcal{C}^0\big([0,1], \mathbb{R}\big)$ equipped with the usual inner product:

$$\varphi(f, g) = \int_0^1 f(t)g(t)\,dt\,.$$

For every nonnegative integer n, let us consider the function $h_n : [0,1] \to \mathbb{R}$, $t \mapsto \cos(2\pi n t)$.

1. Show that the family $(h_n)_{n\in\mathbb{N}}$ is orthogonal.
2. Why does this show that the dimension of E is infinite?
3. Recover this result (the infinite dimension of E) by an other (elementary) method, without using an inner product.

Exercise 8. ***Orthogonal subspace in a space of functions***
Let $E = \mathcal{C}^\infty\big([0,1], \mathbb{R}\big)$, and let us consider the map $\varphi : E \times E \to \mathbb{R}$ defined as:

$$\varphi(f, g) = \int_0^1 \big(f(t)g(t) + f'(t)g'(t)\big)\,dt\,.$$

Let us consider the vector subspaces V and W of E, defined as:

$$V = \big\{f \in E : f(0) = f(1) = 0\big\} \quad \text{and} \quad W = \big\{f \in E : f = f''\big\}\,.$$

1. Prove that φ is an inner product on E.
2. Prove that V and W are orthogonal to one another (meaning: every vector of V is orthogonal to every vector of W, for the inner product above). Can you deduce from this statement that V is included in $W^\perp$? that $V^\perp$ is included in W?
3. Provide an orthogonal basis of W.
4. Prove that $V = W^\perp$.

Exercise 9. ***Distance to a vector subspace of*** $\mathbb{R}^3$
In $\mathbb{R}^3$ equipped with its canonical dot product, let us consider the subset $H = \big\{(x, y, z) \in \mathbb{R}^3 : ax+by+cz = 0\big\}$, where a and b and c are three nonzero real numbers.

1. Why is H a vector subspace of $\mathbb{R}^3$, and what is its dimension?
2. Describe $H^\perp$.
3. Compute the distance between H and a point (x_0, y_0, z_0) of $\mathbb{R}^3$.

Exercise 10. ***Distance to a vector subspace of functions***
Let us consider the space $E = \mathcal{C}^0\big([0,\pi],\mathbb{R}\big)$, equipped with the inner product φ defined as:

$$\varphi(f,g) = \int_0^\pi f(t)g(t)\,dt\,,$$

and let us denote by $\|\cdot\|_2$ the norm associated with φ. Let us denote by id_E the identity function $[0,\pi] \to \mathbb{R}$, $x \mapsto x$, and let $F = \mathrm{span}(\cos,\sin,\mathrm{id}_E)$ in E (here cos and sin stand for the restrictions to the interval $[0,\pi]$ of the functions cosine and sine).

1. Prove that the three functions $(\cos,\sin,\mathrm{id}_E)$ are linearly independent.
2. Write the matrix of φ, restricted to F, in the basis $(\cos,\sin,\mathrm{id}_E)$ of F.
3. Let G denote the subspace of the solutions of the differential equation $\ddot{u} + u = 0$ on the interval $[0,\pi]$.
   a) Prove that G is a vector subspace of F. What is its dimension?
   b) Provide an orthonormal basis (e_1,e_2) of G for the inner product φ.
4. Find a vector e_3 such that (e_1,e_2,e_3) be an orthogonal basis of F for the inner product φ.
5. Let f denote an element of F, and let us write

$$d(f,G) = \inf_{g\in G} \|f-g\|_2\,.$$

   a) Prove that f can be written as: $f = f_0 + g_0$, with g_0 in G and $\varphi(f_0,g_0) = 0$.
   b) Prove that, with this notation, for every g in G,

$$\|f-g\|_2^2 = \|f_0\|_2^2 + \|g-g_0\|_2^2 \quad\text{and}\quad \mathrm{dist}(f,G) = \|f_0\|_2\,.$$

6. Compute the quantity $I = \inf_{(a,b)\in\mathbb{R}^2} \int_0^\pi \big(a\cos(t) + b\sin(t) - t\big)^2\,dt$.

Exercise 11. ***Application of orthogonal projection to minimization***
Let us consider the map $\varphi : \mathbb{R}_2[X]\times\mathbb{R}_2[X] \to \mathbb{R}$, defined as: $\varphi(P,Q) = \int_0^{+\infty} P(t)Q(t)e^{-t}\,dt$.

1. Prove that φ is an inner product on $\mathbb{R}_2[X]$.
2. For every nonnegative integer n, compute the quantity $I_n = \int_0^{+\infty} t^n e^{-t}\,dt$. Provide the matrix of φ in the canonical basis $(1,X,X^2)$ of $\mathbb{R}_2[X]$.
3. Provide an orthonormal basis of the subspace $\mathrm{span}(X,X^2)$ in $\mathbb{R}_2[X]$, for the inner product φ.
4. Compute the quantity $\min_{(a,b)\in\mathbb{R}^2} \int_0^{+\infty} (1-at-bt^2)^2\,e^{-t}\,dt$.

Exercise 12. ***Projection on a finite dimension subspace, general properties***
Let E denote a real vector space equipped with some inner product $\langle\cdot,\cdot\rangle$, and let $\|\cdot\|$ denote the corresponding norm. Let F denote a finite dimensional vector subspace of E, and let p denote the orthogonal projection onto F, in E.

1. Prove that, for every vector x of E,

$$\|p(x)\| \le \|x\| \quad\text{and}\quad \langle x,p(x)\rangle = \|p(x)\|^2\,.$$

2. Provide a necessary and sufficient condition on x so that $p(x) = 0_E$.

Exercise 13. ***Minimization***
Compute the quantities

$$\inf_{(a,b)\in\mathbb{R}^2} \int_0^1 t^2\,\big(\ln(t) - at - b\big)^2\,dt \quad\text{and}\quad \inf_{(a,b)\in\mathbb{R}^2} \int_0^1 t^2\,\big(\sin(t) - at - b\big)^2\,dt\,.$$
